

## ON MUTATIONS OF SELF-INJECTIVE QUIVERS WITH POTENTIAL

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ABSTRACT. We study silting mutations (Okuyama-Rickard complexes) for selfinjective algebras given by quivers with potential (QPs). We show that silting mutation is compatible with QP mutation. As an application, we get a family of derived equivalences of Jacobian algebras.

## 1. INTRODUCTION

Derived categories are nowadays considered as an essential tool in the study of many areas of mathematics. In the representation theory of algebras, derived equivalences of algebras have been one of the central themes and extensively investigated. It is well-known that endomorphism algebras of tilting complexes are derived equivalent to the original algebra [R1]. Therefore it is an important problem to give concrete methods to calculate endomorphism algebras of tilting complexes. In this paper, we focus on one of the fundamental tilting complexes over selfinjective algebras, known as Okuyama-Rickard complexes, which play an important role in the study of Broué's abelian defect group conjecture. From a categorical viewpoint, they are nowadays interpreted as a special case of silting mutation [AI]. We provide a method to determine the quivers with relations of the endomorphism algebras of Okuyama-Rickard complexes when selfinjective algebras are given by quivers with potential (QPs for short).

The notion of QPs was introduced by [DWZ], which plays a significant role in the study of cluster algebras (we refer to [K2]). Recently it has been discovered that mutations of QPs (Definition 2.2) give rise to derived equivalences [BIRS, KeY, M, V]. The aim of this paper is to give a similar (but different) type of derived equivalences by comparing QP mutation and silting mutation (Definition 2.4).

Our main result is the following (see sections 2 and 3 for unexplained notions).

**Theorem 1.1.** (*Proposition 2.7, Theorem 3.1, Corollary 3.2 and Lemma 3.4*) *Let  $(Q, W)$  be a selfinjective QP (Definition 2.1) and  $\Lambda := \mathcal{P}(Q, W)$ . For a set of vertices  $I \subset Q_0$ , we assume the following conditions.*

- *Any vertex in  $I$  is not contained in 2-cycles in  $Q$ .*
- *There are no arrows between vertices in  $I$ .*

(a) *We have an algebra isomorphism*

$$\mathrm{End}_{\mathrm{K}^b(\mathrm{proj} \Lambda)}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q, W)),$$

*where  $\mu_I(\Lambda)$  is left (or right) silting mutation and  $\mu_I(Q, W)$  is QP mutation.*

(b) *If  $\sigma I = \sigma$  for the Nakayama permutation  $\sigma$  of  $\Lambda$ , then  $\mu_I(\Lambda)$  is a tilting complex. In particular,  $\Lambda$  and  $\mathcal{P}(\mu_I(Q, W))$  are derived equivalent.*

Since selfinjective algebras are closed under derived equivalence, we conclude that from (b) above the new QP is also a selfinjective QP, which is a result given in [HI, Theorem

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4.2]. Then we can apply our result to the new QP again and these processes provide a family of derived equivalences. We note that Keller-Yang [KeY] proved that, for two QPs related by QP mutation, their Ginzburg dg algebras, which are certain enhancement of Jacobian algebras, are derived equivalent though their Jacobian algebras are far from being derived equivalent. On the other hand, Theorem 1.1 tells us that Jacobian algebras are already derived equivalent in our setting.

**Notations.** Let  $K$  be an algebraically closed field and  $D := \text{Hom}_K(-, K)$ . All modules are left modules. For a finite dimensional algebra  $\Lambda$ , we denote by  $\text{mod } \Lambda$  the category of finitely generated  $\Lambda$ -modules and by  $\text{add } M$  the subcategory of  $\text{mod } \Lambda$  consisting of direct summands of finite direct sums of copies of  $M \in \text{mod } \Lambda$ . The composition  $fg$  means first  $f$ , then  $g$ . For a quiver  $Q$ , we denote by  $Q_0$  vertices and  $Q_1$  arrows of  $Q$  and by  $a : s(a) \rightarrow e(a)$  the start and end vertices of an arrow or path  $a$ . For a finite dimensional algebra  $KQ/(R)$ , we denote by  $P_i$  the indecomposable projective  $KQ/(R)$ -module corresponding to the vertex  $i \in Q_0$ .

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## 2. PRELIMINARIES

**2.1. Quivers with potential.** We recall the definition of quivers with potential. We follow [DWZ].

- Let  $Q$  be a finite connected quiver without loops. We denote by  $KQ_i$  the  $K$ -vector space with basis consisting of paths of length  $i$  in  $Q$ , and by  $KQ_{i,\text{cyc}}$  the subspace of  $KQ_i$  spanned by all cycles. We denote the *complete path algebra* by

$$\widehat{KQ} = \prod_{i \geq 0} KQ_i$$

and by  $J_{\widehat{KQ}}$  the Jacobson radical of  $\widehat{KQ}$ . A *quiver with potential* (QP) is a pair  $(Q, W)$  consisting of a finite connected quiver  $Q$  without loops and an element  $W \in \prod_{i \geq 2} KQ_{i,\text{cyc}}$ , called a *potential*. For each arrow  $a$  in  $Q$ , the *cyclic derivative*  $\partial_a : \widehat{KQ}_{\text{cyc}} \rightarrow \widehat{KQ}$  is defined as the continuous linear map satisfying  $\partial_a(a_1 \cdots a_d) = \sum_{a_i=a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$  for a cycle  $a_1 \cdots a_d$ . For a QP  $(Q, W)$ , we define the *Jacobian algebra* by

$$\mathcal{P}(Q, W) = \widehat{KQ} / \mathcal{J}(W),$$

where  $\mathcal{J}(W) = \overline{\langle \partial_a W \mid a \in Q_1 \rangle}$  is the closure of the ideal generated by  $\partial_a W$  with respect to the  $J_{\widehat{KQ}}$ -adic topology.

- A QP  $(Q, W)$  is called *trivial* if  $W$  is a linear combination of cycles of length 2 and  $\mathcal{P}(Q, W)$  is isomorphic to the semisimple algebra  $\widehat{KQ}_0$ . It is called *reduced* if  $W \in \prod_{i \geq 3} KQ_{i,\text{cyc}}$ .

Following [HI], we use this terminology.

**Definition 2.1.** We call a QP  $(Q, W)$  *selfinjective* if  $\mathcal{P}(Q, W)$  is a finite dimensional selfinjective algebra.

Next we recall the definition of mutation of QPs.

**Definition 2.2.** For each vertex  $k$  in  $Q$  not lying on a 2-cycle, we define a new QP  $\tilde{\mu}_k(Q, W) := (Q', W')$  as follows.

- (a)  $Q'$  is a quiver obtained from  $Q$  by the following changes.
  - Replace each arrow  $a : k \rightarrow v$  in  $Q$  by a new arrow  $a^* : v \rightarrow k$ .
  - Replace each arrow  $b : u \rightarrow k$  in  $Q$  by a new arrow  $b^* : k \rightarrow u$ .
  - For each pair of arrows  $u \xrightarrow{b} k \xrightarrow{a} v$ , add a new arrow  $[ba] : u \rightarrow v$
- (b)  $W' = [W] + \Delta$  is defined as follows.
  - $[W]$  is obtained from the potential  $W$  by replacing all compositions  $ba$  by the new arrows  $[ba]$  for each pair of arrows  $u \xrightarrow{b} k \xrightarrow{a} v$ .
  - $\Delta = \sum_{\substack{a, b \in Q_1 \\ e(b)=k=s(a)}} [ba]a^*b^*.$

Then *mutation*  $\mu_k(Q, W)$  is defined as a *reduced part* of  $\tilde{\mu}_k(Q, W)$  (we refer to [DWZ]).

**2.2. Silting mutation.** The notion of silting objects was introduced by [KV], which is a generalization of tilting objects. Recently its theory has been rapidly developed and many connections have been discovered, for example [BRT, AI, G, KoY]. In this section, we briefly recall their definitions and properties.

Now let  $\Lambda$  be a finite dimensional algebra and  $\mathcal{T} := K^b(\text{proj } \Lambda)$  be the homotopy category of bounded complexes of finitely generated projective  $\Lambda$ -modules.

**Definition 2.3.** Let  $T$  be an object of  $\mathcal{T}$ . We call  $T$  *silting* (respectively, *tilting*) if  $\text{Hom}_{\mathcal{T}}(T, T[i]) = 0$  for any positive integer  $i > 0$  (for any integer  $i \neq 0$ ) and satisfies  $\mathcal{T} = \text{thick}T$ , where  $\text{thick}T$  denote by the smallest thick subcategory of  $\mathcal{T}$  containing  $T$ .

We call a morphism  $f : X \rightarrow Y$  *left minimal* if any morphism  $g : Y \rightarrow Y$  satisfying  $fg = f$  is an isomorphism. For a object  $M \in \mathcal{T}$ , we call a morphism  $f : X \rightarrow M'$  *left (add  $M$ )-approximation* of  $X$  if  $M'$  belongs to  $\text{add } M$  and  $\text{Hom}_{\mathcal{T}}(f, M'')$  is surjective for any object  $M''$  in  $\text{add } M$ . Dually we define a *right minimal* morphism and a *right (add  $M$ )-approximation*.

**Definition 2.4.** Let  $T$  be a basic silting object in  $\mathcal{T}$  and take an arbitrary decomposition  $T = X \oplus M$ . We take a minimal left (add  $M$ )-approximation  $f : X \rightarrow M'$  of  $X$  and a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1].$$

We put  $\mu_X(T) := Y \oplus M$  and call it a *left silting mutation* of  $T$  with respect to  $X$ . Dually we define a *right silting mutation*.

We recall an important result of silting mutation.

**Theorem 2.5.** [AI, Theorem 2.31] *Any mutation of a silting object is again a silting object.*

Next we give some notations for our setting.

Let  $Q$  be a finite connected quiver and  $\Lambda := KQ/(R)$  be a finite dimensional algebra. We denote by  $\{e_k \mid k \in Q_0\}$  a complete set of primitive orthogonal idempotents of  $\Lambda$ . Take a set of vertices  $I := \{k_1, \dots, k_n\} \subset Q_0$  and we denote by  $e_I := e_{k_1} + \dots + e_{k_n}$  and  $\mu_I(\Lambda) := \mu_{\Lambda e_I}(\Lambda)$ . We remark that an Okuyama-Rickard complex is nothing but a silting object of  $\mathcal{T}$  [AI, Theorem 2.50].

By Theorem 2.5,  $\mu_I(\Lambda)$  is always a silting object of  $\mathcal{T}$ , but it is not necessarily a tilting object. However, for selfinjective algebras, it is a tilting object if it satisfies a condition given by Nakayama permutations.

**Definition 2.6.** Let  $\Lambda$  be a selfinjective algebra above. Then there exists a permutation  $\sigma : Q_0 \rightarrow Q_0$  satisfying  $D(e_k \Lambda) \cong \Lambda e_{\sigma(k)}$  for any  $k \in Q_0$ , where  $\nu := D \operatorname{Hom}_{\Lambda}(-, \Lambda) : \operatorname{mod} \Lambda \rightarrow \operatorname{mod} \Lambda$  is the Nakayama functor. We call  $\sigma$  the *Nakayama permutation* of  $\Lambda$ .

Note that  $\Lambda e_I \cong \nu(\Lambda e_I)$  if and only if  $I = \sigma I$ . The following easy result is useful. We refer to [AH, AI] for the proof.

**Proposition 2.7.** *Let  $\Lambda$  be a selfinjective algebra above. Then  $\mu_I(\Lambda)$  is a tilting object in  $\mathcal{T}$  if and only if  $I = \sigma I$ .*

### 3. MAIN RESULTS

For a set of vertices  $I := \{k_1, \dots, k_n\} \subset Q_0$ , we assume the following conditions.

- (a1) Any vertex in  $I$  is not contained in 2-cycles in  $Q$ .
- (a2) There are no arrows between vertices in  $I$ .

In this case, since the mutation is independent of the choice of order of mutations, we can define the successive mutation

$$\mu_I(Q, W) := \mu_{k_1} \circ \dots \circ \mu_{k_n}(Q, W).$$

Then our main result is the following.

**Theorem 3.1.** *Let  $(Q, W)$  be a selfinjective QP and  $\Lambda := \mathcal{P}(Q, W)$ . Let  $I$  be a set of vertices of  $Q_0$  satisfying the conditions (a1) and (a2). Then we have a  $K$ -algebra isomorphism*

$$\operatorname{End}_{K^b(\operatorname{proj} \Lambda)}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q, W)).$$

We will give the proof in the next section. Combining with Theorem 2.7, we have the following result.

**Corollary 3.2.** *Let  $I$  be a set of vertices of  $Q_0$  satisfying  $\sigma I = I$  and the conditions (a1) and (a2). Then  $\mathcal{P}(Q, W)$  and  $\mathcal{P}(\mu_I(Q, W))$  are derived equivalent.*

*Proof.* By Theorem 2.7,  $\mu_I(\Lambda)$  is a tilting object of  $\mathcal{T}$ . Then  $\Lambda$  and  $\operatorname{End}_{K^b(\operatorname{proj} \Lambda)}(\mu_I(\Lambda))$  are derived equivalent [R1] and the result follows from Theorem 3.1  $\square$

Moreover, since selfinjectivity is preserved by derived equivalence [AR], we have the following result, which is given in [HI, Theorem 4.2].

**Corollary 3.3.** *Let  $I$  be a set of vertices of  $Q_0$  satisfying  $\sigma I = I$  and the conditions (a1) and (a2). Then  $\mu_I(Q, W)$  is a selfinjective QP.*

We note that the Nakayama permutation of  $\mu_I(Q, W)$  is again given by the same permutation [HI, Proposition 4.4.(b)]. By this corollary, we can apply Corollary 3.2 to new QPs repeatedly and, consequently, obtain a lot of derived equivalences.

We considered only left mutation, but the following lemma shows that the same result holds for right mutation.

**Lemma 3.4.** *Under the assumption of Theorem 3.1, we have a  $K$ -algebra isomorphism*

$$\operatorname{End}_{K^b(\operatorname{proj} \Lambda)}(\mu_I(\Lambda)) \cong \operatorname{End}_{K^b(\operatorname{proj} \Lambda)}(\mu'_I(\Lambda)),$$

where  $\mu'_I(\Lambda)$  is a right mutation of  $\Lambda$ .

*Proof.* First, note that we have  $\Lambda^{\text{op}} \cong \mathcal{P}(Q, W)^{\text{op}} \cong \mathcal{P}(Q^{\text{op}}, W^{\text{op}})$ , where  $W^{\text{op}}$  is the corresponding potential of  $Q^{\text{op}}$  to  $W$ . Then it is clear that  $\mathcal{P}(\mu_I(Q^{\text{op}}, W^{\text{op}})) \cong \mathcal{P}(\mu_I(Q, W))^{\text{op}}$  holds.

On the other hand, since  $\Lambda$  is selfinjective, we have a duality

$$\text{Hom}_{\Lambda}(-, \Lambda) : \mathbf{K}^b(\text{proj } \Lambda) \xrightarrow{\sim} \mathbf{K}^b(\text{proj } \Lambda^{\text{op}}),$$

which sends  $\mu'_I(\Lambda)$  to  $\mu_I(\Lambda^{\text{op}})$ . Thus, we have

$$\begin{aligned} \text{End}_{\mathbf{K}^b(\text{proj } \Lambda)}(\mu'_I(\Lambda)) &\cong (\text{End}_{\mathbf{K}^b(\text{proj } \Lambda^{\text{op}})}(\mu_I(\Lambda^{\text{op}})))^{\text{op}} \\ &\cong (\mathcal{P}(\mu_I(Q^{\text{op}}, W^{\text{op}})))^{\text{op}} \quad (\text{Theorem 3.1 to } \Lambda^{\text{op}}) \\ &\cong \mathcal{P}(\mu_I(Q, W)). \end{aligned}$$

□

By this result, we consider only left mutations in this paper.

**Example 3.5.** Let  $(Q, W)$  be the QP given as follows

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow a_1 & & \nwarrow a_6 & \\ & 2 & & 6 & \\ & \swarrow a_2 & & \nwarrow a_5 & \\ 3 & \xrightarrow{a_3} & 4 & \xrightarrow{a_4} & 5, \end{array} \quad W = a_1 a_2 a_3 a_4 a_5 a_6.$$

Then  $(Q, W)$  is a selfinjective QP with a Nakayama permutation  $(153)(264)$ . Let  $\Lambda := \mathcal{P}(Q, W)$  and  $\mathcal{T} := \mathbf{K}^b(\text{proj } \Lambda)$  and take a silting object in  $\mathcal{T}$

$$\mu_1(\Lambda) = \begin{cases} \Lambda e_1 & \xrightarrow{a_1} & \begin{matrix} 0 \\ \Lambda e_2 \\ \oplus \\ \Lambda(1 - e_1) \end{matrix} \end{cases}$$

Then by Theorem 3.1, we have an isomorphism

$$\text{End}_{\mathcal{T}}(\mu_1(\Lambda)) \cong \mathcal{P}(\mu_1(Q, W)),$$

where  $\mu_1(Q, W)$  is the QP given as follows

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow a_1^* & & \nwarrow a_6^* & \\ & 2 & \xleftarrow{[a_6 a_1]} & 6 & \\ & \swarrow a_2 & & \nwarrow a_5 & \\ 3 & \xrightarrow{a_3} & 4 & \xrightarrow{a_4} & 5, \end{array} \quad [a_6 a_1] a_1^* a_6^* + [a_6 a_1] a_2 a_3 a_4 a_5.$$

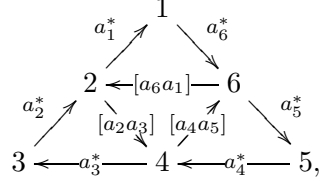
Next we consider the  $\sigma$ -orbit of the vertex 1 and let  $I = \{1, 3, 5\}$ . Then we have a tilting object

$$\mu_I(\Lambda) = \begin{cases} \Lambda e_1 \oplus \Lambda e_3 \oplus \Lambda e_5 & \xrightarrow{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_5 \end{pmatrix}} & \begin{matrix} 0 \\ \Lambda e_2 \oplus \Lambda e_4 \oplus \Lambda e_6 \\ \oplus \\ \Lambda(1 - e_I) \end{matrix} \end{cases}$$

Then we have an isomorphism

$$\text{End}_{\mathcal{T}}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q, W)),$$

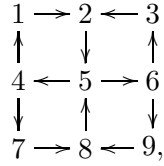
where  $\mu_I(Q, W)$  is the QP given as follows



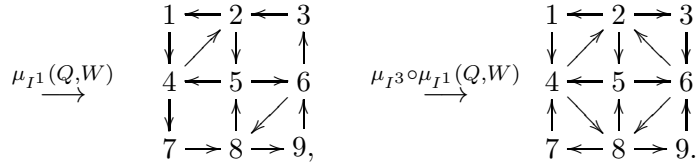
$$[a_6 a_1] a_1^* a_6^* + [a_2 a_3] a_3^* a_2^* + [a_4 a_5] a_5^* a_4^* + [a_6 a_1][a_2 a_3][a_4 a_5].$$

We note that, although  $\mathcal{P}(\mu_I(Q, W))$  is selfinjective and derived equivalent to  $\mathcal{P}(Q, W)$ ,  $\mathcal{P}(\mu_1(Q, W))$  is neither selfinjective nor derived equivalent to  $\mathcal{P}(Q, W)$ .

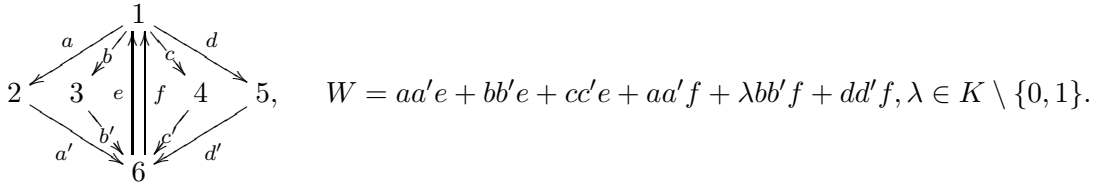
**Example 3.6.** Let  $(Q, W)$  be the QP given as follows



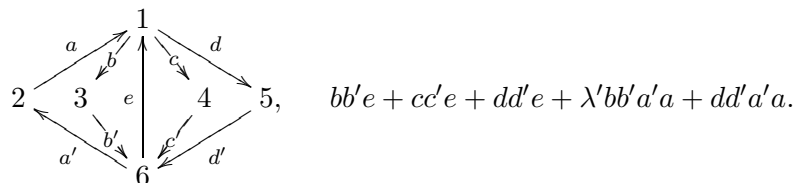
where the potential is the sum of each small squares. Then  $(Q, W)$  is a selfinjective QP with a Nakayama permutation  $(19)(28)(37)(46)(5)$ . For  $\sigma$ -orbits  $I^1 := \{1, 9\}$  and  $I^3 := \{3, 7\}$ , we have selfinjective QPs  $\mu_{I^1}(Q, W)$  and  $\mu_{I^3} \circ \mu_{I^1}(Q, W)$  and their Jacobian algebras are derived equivalent to  $\mathcal{P}(Q, W)$ .



**Example 3.7.** Let  $(Q, W)$  be the QP associated with tubular algebra of type  $(2, 2, 2, 2)$



Then  $(Q, W)$  is a selfinjective QP [HI] and the Nakayama permutation is the identity. Thus mutation of the QP at any vertex admits a derived equivalence in this case. For example,  $\mu_2(Q, W)$  is the following QP with  $\lambda' = \frac{\lambda}{\lambda-1}$



Thus  $\mu_2(Q, W)$  is a selfinjective QP and  $\mathcal{P}(\mu_2(Q, W))$  is derived equivalent to  $\mathcal{P}(Q, W)$ .

Thus, from a given selfinjective Jacobian algebra, QP mutations give new selfinjective Jacobian algebra which are derived equivalent to the original one. Here we give a natural question that we find important for better understanding of QPs and derived equivalences.

## 4. PROOF OF MAIN RESULT

#### 4.1. 2-almost split sequences.

$$U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X$$
$$\mathcal{T}(T, U_1) \xrightarrow{f_1} \mathcal{T}(T, U_0) \xrightarrow{f_0} J_{\mathcal{T}}(T, X) \rightarrow 0$$
$$X \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0$$
$$\mathcal{T}(U_0, T) \xrightarrow{f_1} \mathcal{T}(U_1, T) \xrightarrow{f_2} J_{\mathcal{T}}(X, T) \rightarrow 0$$
$$X \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X$$

a weak 2-almost split sequence in  $\text{add } T$  if  $U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X$  is a right 2-almost split sequence and  $X \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0$  is a left 2-almost split sequence.

Let  $Q$  be a finite connected quiver. For  $a \in Q_1$ , define a *right derivative*  $\partial_a^r : J_{\widehat{KQ}} \rightarrow \widehat{KQ}$  and a *left derivative*  $\partial_a^l : J_{\widehat{KQ}} \rightarrow \widehat{KQ}$  by

$$\begin{aligned}\partial_a^r(a_1 a_2 \cdots a_{m-1} a_m) &= \begin{cases} a_1 a_2 \cdots a_{m-1} & \text{if } a_m = a, \\ 0 & \text{otherwise,} \end{cases} \\ \partial_a^l(a_1 a_2 \cdots a_{m-1} a_m) &= \begin{cases} a_2 \cdots a_{m-1} a_m & \text{if } a_1 = a, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

and extend to  $J_{\widehat{KQ}}$  linearly and continuously.

We call an element of  $\widehat{KQ}$  *basic* if it is a formal linear sum of paths in  $Q$  with a common start and a common end. Then we have the following result.

**Proposition 4.2.** [BIRS, Proposition 3.1, 3.3] *Let  $Q$  be a finite connected quiver and  $\Gamma$  be a basic finite dimensional algebra. Let  $\phi : \widehat{KQ} \rightarrow \Gamma$  be an algebra homomorphism and  $R$  be a finite set of basic elements in  $J_{\widehat{KQ}}$ . Then the following conditions are equivalent.*

- (a)  $\phi$  is surjective and  $\text{Ker } \phi = \overline{I}$  for the ideal  $I = \langle R \rangle$  of  $\widehat{KQ}$ , where  $\overline{(\ )}$  denote the closure.
- (b) The following sequence is exact for any  $i \in Q_0$ .

$$\bigoplus_{r \in R, e(r)=i} \Gamma(\phi s(r)) \xrightarrow{r(\phi \partial_a^r r)_a} \bigoplus_{a \in Q_1, e(a)=i} \Gamma(\phi s(a)) \xrightarrow{a(\phi a)} J_\Gamma(\phi i) \longrightarrow 0.$$

**4.2. Our settings.** We keep the assumption of Theorem 3.1. Without loss of generality, we can assume that  $(Q, W)$  is reduced since  $\mathcal{P}(\mu_I(Q, W))$  is isomorphic to  $\mathcal{P}(\mu_I(Q_{\text{red}}, W_{\text{red}}))$ , where  $(Q_{\text{red}}, W_{\text{red}})$  is a reduced part of  $(Q, W)$  ([DWZ]).

For a pair of arrows  $a$  and  $b$ , define  $\partial_{(a,b)} W$  by

$$\partial_{(a,b)}(a_1 a_2 \cdots a_m) = \sum_{a_i=a, a_{i+1}=b} a_{i+2} \cdots a_m a_1 \cdots a_{i-1}$$

for any cycle  $a_1 \cdots a_m$  in  $W$  and extend linearly and continuously. We denote by  $\phi$  the natural surjective map  $\widehat{KQ} \rightarrow \mathcal{P}(Q, W)$ . We simply denote  $\phi p$  by  $p$  for any morphism  $p$  in  $\widehat{KQ}$ . Then, for any  $i \in Q_0$ , we have the following exact sequence in  $\text{mod } \Lambda$  [HI, Theorem 3.7].

$$(1) \quad P_i \xrightarrow{f_{i2} := (b)_b} \overbrace{\bigoplus_{b \in Q_1, s(b)=i} P_{e(b)}}^{U_{i1} :=} \xrightarrow{f_{i1} := b(\partial_{(a,b)} W)_a} \overbrace{\bigoplus_{a \in Q_1, e(a)=i} P_{s(a)}}^{U_{i0} :=} \xrightarrow{f_{i0} := a(a)} P_i.$$

Note that  $f_{i2}$  is a minimal left  $(\text{add}(\Lambda/P_i))$ -approximation and  $f_{i0}$  is a minimal right  $(\text{add}(\Lambda/P_i))$ -approximation. We embed the morphism  $f_{i2}$  to a triangle in  $\mathcal{T}$

$$(2) \quad P_i \xrightarrow{f_{i2}} U_{i1} \xrightarrow{h_i} P_i^* \rightarrow P_i[1].$$

Then  $P_i^*$  is the object

$$(\cdots \rightarrow 0 \rightarrow P_i \xrightarrow{-1} P_i \xrightarrow{f_{i2}} U_{i1} \rightarrow 0 \rightarrow \cdots)$$

and we have a complex

$$(3) \quad P_i^* \xrightarrow{g_i} U_{i0} \xrightarrow{f_{i0}} P_i$$



in  $\mathcal{T}$ , where  $g_i = (g_i^j)_{j \in \mathbb{Z}}$  is defined by  $g_i^0 = f_{i1}$  and  $g_i^j = 0$  for  $j \neq 0$ .

Let  $(Q', W') := \tilde{\mu}_I(Q, W) = \tilde{\mu}_{k_1} \circ \cdots \circ \tilde{\mu}_{k_n}(Q, W)$  and  $W' = [W] + \Delta$  is defined as follows.

- $[W]$  is obtained from the potential  $W$  by replacing all compositions  $b_i a_i$  by the new arrows  $[b_i a_i]$  for each pair of arrows  $u_i \xrightarrow{b_i} k_i \xrightarrow{a_i} v_i$  and  $k_i \in I$ .

- $\Delta = \sum_{\substack{a_i, b_i \in Q_1, k_i \in I \\ e(b_i) = k_i = s(a_i)}} [b_i a_i] a_i^* b_i^*.$

Then, we define a  $K$ -algebra homomorphism  $\phi' : \widehat{KQ'} \rightarrow \text{End}_{\mathcal{T}}(T)$  as follows.

- $\phi'$  is the same as  $\phi$  on  $Q \cap Q'$ .
- For any  $l \in I$ , let  $\phi'[ab] = (\phi a)(\phi b)$  for each pair of arrows  $a: i \rightarrow l$  and  $b: l \rightarrow j$  in  $Q$ .
- For any  $l \in I$ , let  $\phi' l = P_l^*$ , and  $\phi' a^*$  ( $e(a) = l$ ) and  $\phi' b^*$  ( $s(b) = l$ ) are defined by the equalities

$$\begin{aligned} \phi'((a^*)_{a \in Q_1, e(a)=l}) &= g_l \in \mathcal{T}(P_l^*, \bigoplus_{a \in Q_1, e(a)=l} P_{s(a)}), \\ \phi'((b^*)_{b \in Q_1, s(b)=l}) &= -h_l \in \mathcal{T}(\bigoplus_{b \in Q_1, s(b)=l} P_{e(b)}, P_l^*), \end{aligned}$$

where  $g_l, h_l$  are given in (2), (3).

As before, we simply denote  $\phi' p$  by  $p$  for any morphism  $p$  in  $\widehat{KQ'}$ . Then we give the following proposition.

**Proposition 4.3.** *For a vertex  $l$  in  $Q_0$  with  $l \in I$ , we have the following right 2-almost split sequence in  $\text{add } T$*

$$\bigoplus_{a \in Q_1, e(a)=l} P_{s(a)} \xrightarrow{a([ab])_b} \bigoplus_{b \in Q_1, s(b)=l} P_{e(b)} \xrightarrow{b(b^*)} P_l^*.$$

Next take a vertex  $i \in Q_0$  with  $i \notin I$ . Let  $I_1 := \{l \in I \mid \exists u \in Q_1; s(u) = l, e(u) = i\}$  and  $I_2 := \{l \in I \mid \exists v \in Q_1; s(v) = i, e(v) = l\}$ .

We define complexes by

$$\begin{aligned} (P_{I_1}^* \xrightarrow{a^*} U_{I_1 0} \xrightarrow{a} P_{I_1}) &:= \bigoplus_{k_1 \in I_1} (P_{k_1}^* \xrightarrow{(a^*)_{a}} U_{k_1 0} \xrightarrow{a(a)} P_{k_1}), \\ (P_{I_2} \xrightarrow{b} U_{I_2 1} \xrightarrow{b^*} P_{I_2}^*) &:= \bigoplus_{k_2 \in I_2} (P_{k_2} \xrightarrow{(b)_b} U_{k_2 1} \xrightarrow{b(b^*)} P_{k_2}^*). \end{aligned}$$

Moreover we decompose  $U_{i0} = P_{I_1} \oplus U'_{i0}$  and  $U_{i1} = P_{I_2} \oplus U'_{i1}$ . Then we can write the sequence (1) by

$$P_i \xrightarrow{(d \ v)} U'_{i1} \oplus P_{I_2} \xrightarrow{\begin{pmatrix} f_1 a & f'_1 \\ b_{f_2} a & b_{f'_2} \end{pmatrix}} P_{I_1} \oplus U'_{i0} \xrightarrow{(u \ c)} P_i,$$

where

$$\begin{aligned}
\mathbf{u} &:= u(u) \text{ for } \{u \in Q_1 \mid s(u) \in I, e(u) = i\}, \\
\mathbf{c} &:= c(c) \text{ for } \{c \in Q_1 \mid s(c) \notin I, e(c) = i\}, \\
\mathbf{v} &:= (v)_v \text{ for } \{v \in Q_1 \mid s(v) = i, e(v) \in I\}, \\
\mathbf{d} &:= (d)_d \text{ for } \{d \in Q_1 \mid s(d) = i, e(d) \notin I\}, \\
f_1 &:= d(\partial_{([au], d)}[W])_{(a, u)}, \quad f'_1 := d(\partial_{(c, d)}[W])_c, \\
f_2 &:= (v, b)(\partial_{([au], [vb])}[W])_{(a, u)}, \quad f'_2 := (v, b)(\partial_{(c, [vb])}[W])_c.
\end{aligned}$$

Then we have the following diagram

$$\begin{array}{ccccccc}
P_{I_1}^* & \xrightarrow{\mathbf{a}^*} & U_{I_1 0} & \xrightarrow{\mathbf{a}} & P_{I_1} & & \\
& \nearrow f_1 & \uparrow & & \searrow \mathbf{u} & & \\
P_i & \xrightarrow{\mathbf{d}} & U'_{i1} & \xrightarrow{f'_1} & U'_{i0} & \xrightarrow{\mathbf{c}} & P_i \\
& \searrow \mathbf{v} & & \downarrow f_2 & & \nearrow f'_2 & \\
& & P_{I_2} & \xrightarrow{\mathbf{b}} & U_{I_2 1} & \xrightarrow{\mathbf{b}^*} & P_{I_2}^* \\
& & & & \nearrow & & \searrow \mathbf{v}^*
\end{array}$$

where  $\mathbf{v}^* = v(v^*)$  for  $\{v \in Q_1 \mid s(v) = i, e(v) \in I\}$ . Then we give the following proposition.

**Proposition 4.4.** *For a vertex  $i$  in  $Q_0$  with  $i \notin I$ , we have the following right 2-almost split sequence in  $\text{add } T$ .*

$$P_{I_1}^* \oplus U'_{i1} \oplus U_{I_2 1} \xrightarrow{\begin{pmatrix} \mathbf{a}^* & 0 & 0 \\ f_1 & f'_1 & 0 \\ f_2 & f'_2 & \mathbf{b}^* \end{pmatrix}} U_{I_1 0} \oplus U'_{i0} \oplus P_{I_2}^* \xrightarrow{\begin{pmatrix} \mathbf{au} \\ \mathbf{c} \\ \mathbf{v}^* \end{pmatrix}} P_i.$$

Before starting to prove Propositions 4.3, 4.4, we prove Theorem 3.1 by using them.

*Proof of Theorem 3.1.* By Proposition 4.2, we can determine the quiver and relations of  $\text{End}_{\mathcal{T}}(T)$  by obtaining the projective presentations of simple  $\text{End}_{\mathcal{T}}(T)$ -modules. They are given by Propositions 4.3 and 4.4 from the definition of a right 2-almost split sequence.  $\square$

The rest of this paper is devoted to showing Propositions 4.3 and 4.4.

**4.3. Exactness of some sequences.** We keep the notation of previous subsections. We will show the following proposition.

**Proposition 4.5.** (a) *For any  $l \in I$ , we have the following weak 2-almost split sequence in  $\text{add } T$ :*

$$P_l^* \xrightarrow{g_l} U_{l0} \xrightarrow{f_{l0}f_{l2}} U_{l1} \xrightarrow{h_l} P_l^*,$$

where  $g_l, f_{l0}, f_{l2}$  and  $h_l$  are the morphism given in (1), (2), (3).

(b) *For any  $l, m \in I$ , the following sequence is exact:*

$$\mathcal{T}(P_m^*, P_l^*) \xrightarrow{g_l} \mathcal{T}(P_m^*, U_{l0}) \xrightarrow{f_{l0}} \mathcal{T}(P_m^*, P_l).$$

Therefore the following sequence is exact:

$$\mathcal{T}(T, P_{I_1}^*) \xrightarrow{\mathbf{a}^*} \mathcal{T}(T, U_{I_1 0}) \xrightarrow{\mathbf{a}} \mathcal{T}(T, P_{I_1}).$$

(c) For any  $i \in Q_0$  with  $i \notin I$  and any  $l \in I$ , the following sequence is exact:

$$\mathcal{T}(P_l^*, U_{i1}) \xrightarrow{f_{i1}} \mathcal{T}(P_l^*, U_{i0}) \xrightarrow{f_{i0}} \mathcal{T}(P_l^*, P_i).$$

Therefore the following sequence is exact:

$$\mathcal{T}(T, U_{i1}) \xrightarrow{f_{i1}} \mathcal{T}(T, U_{i0}) \xrightarrow{f_{i0}} \mathcal{T}(T, P_i).$$

We prepare the following notations. For any  $l \in I$ , we denote by  $C_l := \text{Coker } f_{l2}$ . Then we have the following exact sequence

$$P_l \xrightarrow{f_{l2}} U_{l1} \xrightarrow{q_l} C_l \rightarrow 0 \quad \text{and} \quad 0 \rightarrow C_l \xrightarrow{r_l} U_{l0} \xrightarrow{f_{l0}} P_l.$$

First we give the following easy observation.

**Lemma 4.6.** *Let  $p = (p^i)_{i \in \mathbb{Z}} \in \mathcal{T}(P_l^*, P_m^*)$  be a morphism for  $l, m \in I$ . Then we have the following commutative diagram.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Soc } P_l & \xrightarrow{d_l} & P_l & \xrightarrow{f_{l2}} & U_{l1} & \xrightarrow{q_l} & C_l & \longrightarrow & 0 \\ & & \downarrow p^{-2} & & \downarrow p^{-1} & & \downarrow p^0 & & \downarrow p^1 & & \\ 0 & \longrightarrow & \text{Soc } P_m & \xrightarrow{d_m} & P_m & \xrightarrow{f_{m2}} & U_{m1} & \xrightarrow{q_m} & C_m & \longrightarrow & 0. \end{array}$$

If  $p \in J_{\mathcal{T}}(P_l^*, P_m^*)$ , then  $p^{-2} = 0$ , and there exist  $j_{lm}^0 \in \text{Hom}_{\Lambda}(U_{l1}, P_m)$  and  $j_{lm}^1 \in \text{Hom}_{\Lambda}(C_l, U_{m1})$  such that  $p^{-1} = f_{l2}j_{lm}^0$ ,  $p^0 = j_{lm}^0 f_{m2} + q_l j_{lm}^1$  and  $p^1 = j_{lm}^1 q_m$ .

*Proof.* The first assertion is clear. Assume that  $p \in J_{\mathcal{T}}(P_l^*, P_m^*)$ . If  $p^{-2}$  is isomorphic, then it implies that  $p^{-1}$  is isomorphic since  $d_l$  is an injective hull. Since  $f_{l2}$  and  $f_{m2}$  are minimal left  $(\text{add}(\Lambda/P_l))$ -approximation,  $p^0$  is also isomorphic, a contradiction to  $p \in J_{\mathcal{T}}(P_l^*, P_m^*)$ . Thus  $p^{-2}$  is not an isomorphism. Because  $\text{Soc } P_l$ ,  $\text{Soc } P_m$  are simple modules, we have  $p^{-2} = 0$ . Therefore we obtain  $d_l p^{-1} = p^{-2} d_m = 0$ . Since  $P_m$  is an injective module, there exists  $j_{lm}^0 \in \text{Hom}_{\Lambda}(U_{l1}, P_m)$  such that  $p^{-1} = f_{l2}j_{lm}^0$ . Then since  $U_{m1}$  is an injective module and  $f_{l2}(p^0 - j_{lm}^0 f_{m2}) = 0$ , there exists  $j_{lm}^1 \in \text{Hom}_{\Lambda}(C_l, U_{m1})$  such that  $q_l j_{lm}^1 = p^0 - j_{lm}^0 f_{m2}$ . Since  $q_l$  is surjective, we have  $p^1 = j_{lm}^1 q_m$ .  $\square$

Now we are ready to prove Proposition 4.5.

*Proof.* (a) (i) We will show that  $h_l$  is right almost split in  $\text{add } T$ .

Take any morphism  $p = (p^i)_{i \in \mathbb{Z}} \in J_{\mathcal{T}}(\Lambda/P_l, P_l^*)$ . Then clearly  $p^0 \in \text{Hom}_{\Lambda}(\Lambda/P_l, U_{l0})$  gives a morphism  $g = (g^i)_{i \in \mathbb{Z}} \in \mathcal{T}(\Lambda/P_l, U_{l0})$  by  $g^0 = p^0$  and  $g^i = 0$  for  $i \neq 0$ . Thus we have  $gh_l = p$ .

Next for any  $m \in I$ , take any morphism  $p = (p^i)_{i \in \mathbb{Z}} \in J_{\mathcal{T}}(P_m^*, P_l^*)$ . By Lemma 4.6, there exists  $j_{ml}^0 \in \text{Hom}_{\Lambda}(U_{m1}, P_l)$  such that  $p^{-1} = f_{m2}j_{ml}^0$ . Then the morphism  $p^0 - j_{ml}^0 f_{l2} \in \text{Hom}_{\Lambda}(U_{m1}, U_{l1})$  gives a morphism  $g \in \mathcal{T}(P_m^*, U_{l1})$  satisfying  $p = gh_l$ .

$$\begin{array}{ccccc} & & & U_{m1} & \\ & & & \uparrow f_{m2} & \\ & & & P_m & \\ & & & \downarrow p^{-1} & \\ & & & U_{l1} & \\ & & & \uparrow j_{ml}^0 & \\ & & & P_l & \\ & & & \downarrow f_{l2} & \\ & & & 0 & \\ & & & \uparrow & \\ & & & U_{l1} & \\ & & & \downarrow \text{id} & \\ & & & 0 & \end{array}$$

Detailed description: The diagram shows a complex set of relationships between modules. At the top is  $U_{m1}$ , which maps to  $P_m$  via  $f_{m2}$ .  $P_m$  maps to  $U_{l1}$  via  $p^{-1}$ .  $U_{l1}$  maps to  $P_l$  via  $j_{ml}^0$ .  $P_l$  maps to  $0$  via  $f_{l2}$ .  $0$  maps to  $U_{l1}$  via an inclusion map.  $U_{l1}$  maps to  $U_{l0}$  via  $\text{id}$ .  $U_{l0}$  maps to  $0$  via  $p^0 - j_{ml}^0 f_{l2}$ .  $U_{l0}$  also maps to  $P_l$  via  $p^{-2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps 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maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  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.  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via  $q_l$ .  $C_l$  maps to  $0$  via  $p^1$ .  $0$  maps to  $C_l$  via  $r_l$ .  $C_l$  maps to  $U_{l0}$  via  $f_{l0}$ .  $U_{l0}$  maps to  $P_l$  via  $f_{l2}$ .  $P_l$  maps to  $U_{l1}$  via  $d_l$ .  $U_{l1}$  maps to  $C_l$  via

(ii) We will show that  $f_{l0}f_{l2}$  is a pseudo-kernel of  $h_l$  in  $\text{add } T$ .

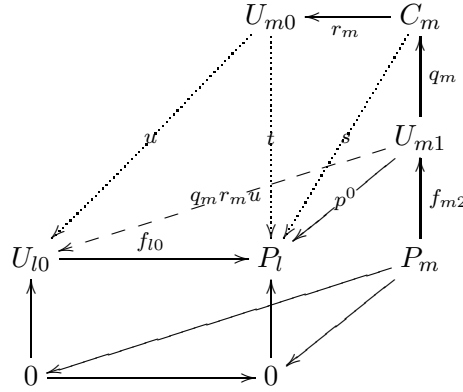
Since (2) is a triangle, we have an exact sequence  $\mathcal{T}(T, P_l) \xrightarrow{f_{l2}} \mathcal{T}(T, U_{l1}) \xrightarrow{h_l} \mathcal{T}(T, P_l^*)$ .

Thus we only have to show that  $\mathcal{T}(T, U_{l0}) \xrightarrow{f_{l0}} \mathcal{T}(T, P_l)$  is surjective.

Take any morphism  $p = (p^i)_{i \in \mathbb{Z}} \in \mathcal{T}(\Lambda/P_I, P_l)$ . Then since  $f_{l0}$  is a right  $(\text{add}(\Lambda/P_I))$ -approximation, there exists  $g^0 \in \text{Hom}_\Lambda(\Lambda/P_I, U_{l0})$  such that  $p^0 = g^0 f_{l0}$ . Thus  $g^0$  gives a morphism  $g \in \mathcal{T}(\Lambda/P_I, U_{l0})$  satisfying  $p = g f_{l0}$ .

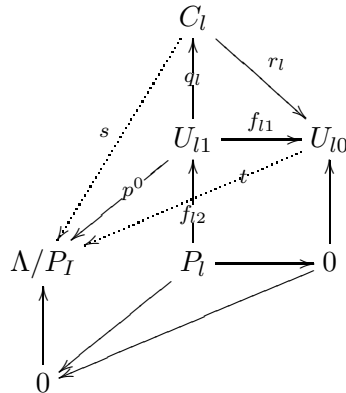
Next for any  $m \in I$ , take any morphism  $p = (p^i)_{i \in \mathbb{Z}} \in \mathcal{T}(P_m^*, P_l)$ . Then by  $f_{m2}p^0 = 0$ , there exists  $s \in \text{Hom}_\Lambda(C_m, P_l)$  such that  $p^0 = q_m s$ .

Then since  $P_l$  is an injective module, there exists  $t \in \text{Hom}_\Lambda(U_{m0}, P_l)$  such that  $s = r_m t$ . Moreover by the assumption (a2), we have  $P_l \notin \text{add } U_{m0}$ . Then since  $f_{l0}$  is a right  $(\text{add}(\Lambda/P_I))$ -approximation, there exists  $u \in \text{Hom}_\Lambda(U_{m0}, U_{l0})$  such that  $t = u f_{l0}$ . Thus we have  $p^0 = (q_m r_m u) f_{l0}$ , and  $g^0 := q_m r_m u$  gives a morphism  $g \in \mathcal{T}(P_m^*, U_{l0})$  satisfying  $p = g f_{l0}$ .

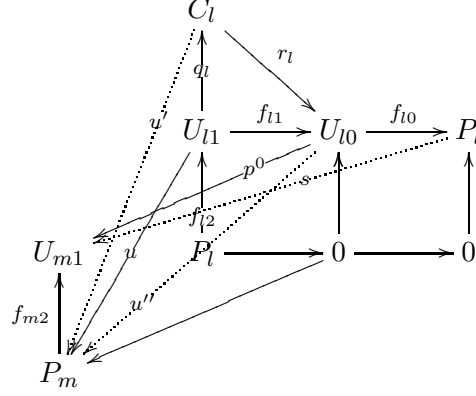


(iii) We will show that  $g_l$  is left almost split in  $\text{add } T$ .

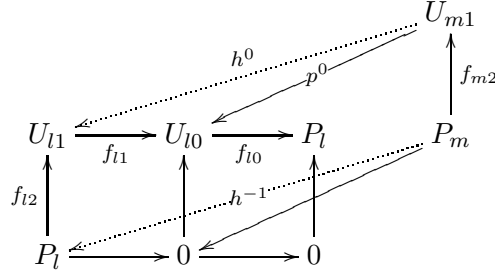
Take any morphism  $p = (p^i)_{i \in \mathbb{Z}} \in J_{\mathcal{T}}(P_l^*, \Lambda/P_I)$ . Then by  $f_{l2}p^0 = 0$ , there exists  $s \in \text{Hom}_\Lambda(C_l, \Lambda/P_I)$  such that  $p^0 = q_l s$ . Then since  $\Lambda/P_I$  is an injective module and  $r_l$  is injective, there exists  $t \in \text{Hom}_\Lambda(U_{l1}, \Lambda/P_I)$  such that  $s = r_l t$ . Then we have  $p^0 = q_l r_l t = f_{l1}t$  and  $t$  gives a morphism  $t \in \mathcal{T}(U_{l1}, \Lambda/P_I)$  satisfying  $p = g_l t$ .



Thus there exists  $s \in \text{Hom}_\Lambda(P_l, U_{m_1})$  such that  $p^0 - u''f_{m_2} = f_{l_0}s$ . By the assumption (a2), we have  $P_l \notin \text{add } U_{m_1}$ . Then since  $f_{l_2} : P_l \rightarrow U_{l_1}$  is a left  $(\text{add}(\Lambda/P_l))$ -approximation, there exists  $t \in \text{Hom}_\Lambda(U_{l_1}, U_{m_1})$  such that  $p^0 - u''f_{m_2} = (f_{l_0}f_{l_2})t$ . Then  $t$  gives a morphism  $t \in \mathcal{T}(U_{l_1}, U_{m_1})$  satisfying  $p = (f_{l_0}f_{l_2})t$ .



(b) Assume that  $p = (p^i)_{i \in \mathbb{Z}} \in \mathcal{T}(P_m^*, U_{l0})$  satisfies  $pf_{l0} = 0$ . Then since  $p^0 f_{l0} = 0$ , there exists  $h^0 \in \text{Hom}_\Lambda(U_{m1}, U_{l1})$  such that  $p^0 = h^0 f_{l1}$ . Since  $f_{m2} h^0 f_{l1} = f_{m2} p^0 = 0$ , there exists  $h^{-1} \in \text{Hom}_\Lambda(P_m, P_l)$  such that  $f_{m2} h^0 = h^{-1} f_{l2}$ . Then  $h^0, h^{-1}$  give a morphism  $h \in \mathcal{T}(P_m^*, P_l^*)$  satisfying  $p = h g_l$ .

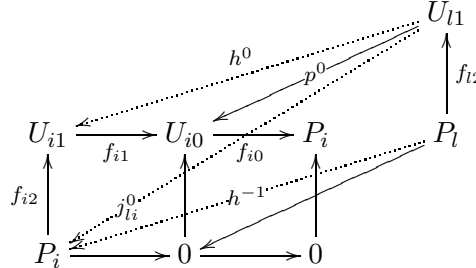


Moreover, it is easy to see that the following sequence is exact by (1)

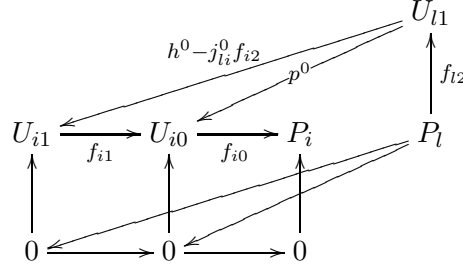
$$\mathcal{T}(\Lambda/P_I, P_l^*) \xrightarrow{g_l} \mathcal{T}(\Lambda/P_I, U_{l0}) \xrightarrow{f_{l0}} \mathcal{T}(\Lambda/P_I, P_l).$$

Hence the second statement follows immediately from the first one.

(c) Assume that  $p = (p^i)_{i \in \mathbb{Z}} \in \mathcal{T}(P_l^*, U_{i0})$  satisfies  $pf_{i0} = 0$ . Then there exists  $h^0 \in \text{Hom}_\Lambda(U_{l1}, U_{i1})$  such that  $p^0 = h^0 f_{i1}$ . Moreover, since  $f_{l2} h^0 f_{i1} = f_{l2} p^0 = 0$ , there exists  $h^{-1} \in \text{Hom}_\Lambda(P_l, P_i)$  such that  $f_{l2} h^0 = h^{-1} f_{i2}$ . Since  $l \neq i$ , we have  $h \in J_{\mathcal{T}}(P_l^*, P_i^*)$ . Then by Lemma 4.6, there exists  $j_{li}^0 \in \text{Hom}_\Lambda(U_{l1}, P_i)$  such that  $h^{-1} = f_{l2} j_{li}^0$ .



Then  $h^0 - j_{li}^0 f_{i2} \in \text{Hom}_\Lambda(U_{l1}, U_{i1})$  gives a morphism  $h \in \mathcal{T}(P_l^*, U_{i1})$  satisfying  $p = h f_{i1}$ .



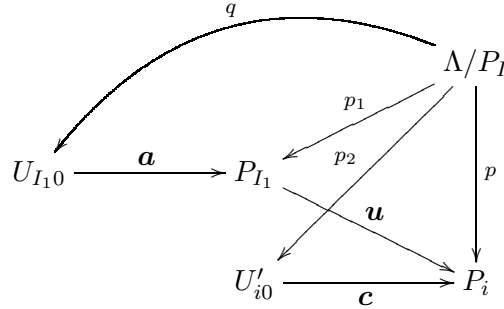
The second statement follows immediately from the first one.  $\square$

**4.4. Proof of Propositions 4.3 and 4.4.** Then we give the proof of Propositions 4.3 and 4.4 and complete the proof of Theorem 3.1.

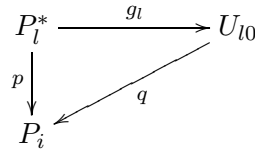
*Proof.* Proposition 4.3 immediately follows from Proposition 4.5 (a). We will show Proposition 4.4. Since we have  $\mathbf{b}^* \mathbf{v}^* + (f_2 \ f_2') \begin{pmatrix} \mathbf{a} \mathbf{u} \\ \mathbf{c} \end{pmatrix} = {}_{b,v}(\mathbf{b}^* \mathbf{v}^* + (\partial_{[vb]} W)) = 0$  by the definition of the algebra homomorphism, it is a complex.

*Step 1.* We will show that  $\begin{pmatrix} \mathbf{a} \mathbf{u} \\ \mathbf{c} \end{pmatrix}$  is right almost split in  $\text{add } T$ .

(i) First we will show that any morphism  $p \in J_{\mathcal{T}}(\Lambda/P_I, P_i)$  factors through  $\begin{pmatrix} \mathbf{a} \mathbf{u} \\ \mathbf{c} \end{pmatrix}$ . Since  $\begin{pmatrix} \mathbf{u} \\ \mathbf{c} \end{pmatrix} : P_{I_1} \oplus U'_{i0} \rightarrow P_i$  is right almost split in  $\text{add } \Lambda$ , there exists  $(p_1 \ p_2) \in \mathcal{T}(\Lambda/P_I, P_{I_1} \oplus U'_{i0})$  such that  $p = p_1 \mathbf{u} + p_2 \mathbf{c}$ . Since  $\mathbf{a} : U_{I_10} \rightarrow P_{I_1}$  is a right  $(\text{add}(\Lambda/P_{I_1}))$ -approximation, there exists  $q \in \mathcal{T}(\Lambda/P_I, U_{I_10})$  such that  $p_1 = q\mathbf{a}$ . Thus we have  $p = (q \ p_2) \begin{pmatrix} \mathbf{a} \mathbf{u} \\ \mathbf{c} \end{pmatrix}$ .



(ii) Next we take any  $p \in J_{\mathcal{T}}(P_l^*, P_i)$  for  $l \in I$  with  $l \notin I_2$ . Note that there is no arrow  $i \rightarrow l$  in  $Q$  in this case. Since  $g_l$  is left almost split in  $\text{add } T$  by Proposition 4.5 (a), there exists  $q \in \mathcal{T}(U_{l0}, P_i)$  such that  $p = g_l q$ . Since  $P_l \notin \text{add } U_{i1}$ , we have  $P_i \notin \text{add } U_{l0}$ . Thus  $q \in J_{\mathcal{T}}(U_{l0}, P_i)$  holds. Moreover by the assumption (a2), we have  $P_{I_1} \notin \text{add } U_{l0}$ . Then the first case implies that  $q$  factors through  $\begin{pmatrix} \mathbf{a} \mathbf{u} \\ \mathbf{c} \end{pmatrix}$ .



(iii) Take  $l \in I_2$  and decompose  $P_{I_2}^* = (P_l^*)^{n_l} \oplus X$  such that  $X \notin \text{add } P_l^*$ . We will show that the map

$$(v^*)_{\{v \in Q \mid v: i \rightarrow l\}} : (\mathcal{T}/J_{\mathcal{T}})(P_l^*, (P_l^*)^{n_l}) \rightarrow (J_{\mathcal{T}}/J_{\text{add } T}^2)(P_l^*, P_i)$$

is bijective. Since  $\mathcal{T}$  is a Krull-schmidt category and  $P_l^*$  is indecomposable, we have  $K = (\mathcal{T}/J_{\mathcal{T}})(P_l^*, P_l^*)$ . On the other hand, by Proposition 4.5 (a), we have that  $g_l :$

$P_l^* \rightarrow U_{l0} = \oplus_{\{a \in Q, e(a)=l\}} P_{s(a)}$  is minimal left almost split in  $\text{add } T$  since the middle morphism  $f_{l0}f_{l2}$  in the sequence of Proposition 4.5 (a) belongs to  $J_{\mathcal{T}}$ . Thus we have that  $(J_{\mathcal{T}}/J_{\text{add } T}^2)(P_l^*, P_i)$  is a  $K$ -vector space with basis  $\{v^* \mid v \in Q_1; v : i \rightarrow l\}$ . Thus the above map is bijective.

Then take any  $p \in J_{\mathcal{T}}(P_l^*, P_i)$  for  $l \in I_2$ . Let  $(v^*)_v : (P_l^*)^{n_l} \rightarrow P_i$  be a restriction of  $v^*$ . By the above bijection, there exists  $p_1 \in \mathcal{T}(P_l^*, (P_l^*)^{n_l})$  such that  $p - p_1(v^*)_v \in J_{\text{add } T}^2(P_l^*, P_i)$ . Since  $g_l$  is right almost split in  $\text{add } T$  by (a), there exists  $q \in J_{\mathcal{T}}(U_{l0}, P_i)$  such that  $p - p_1(v^*)_v = g_l q$ . Then, by the same argument of (i) and (ii),  $q$  factors through  $\begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \\ \mathbf{v}^* \end{pmatrix}$ . Thus  $p$  factors through  $\begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \\ \mathbf{v}^* \end{pmatrix}$ .

$$\begin{array}{ccccc}
 U_{I_1 0} & \xrightarrow{\mathbf{a}} & P_{I_1} & & P_l^* \xrightarrow{g_l} U_{l0} \\
 & & \searrow \mathbf{u} & \nearrow p_1 & \downarrow p \\
 & & U'_{i0} & \xrightarrow{\mathbf{c}} & P_i \\
 & & \nearrow \mathbf{v}^* & & \nwarrow q \\
 & & P_{I_2}^* & & 
 \end{array}$$

*Step 2.* We will show that  $\begin{pmatrix} \mathbf{a}^* & 0 & 0 \\ f_1 & f'_1 & 0 \\ f_2 & f'_2 & \mathbf{b}^* \end{pmatrix}$  is a pseudo-kernel of  $\begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \\ \mathbf{v}^* \end{pmatrix}$  in  $\text{add } T$ .

Assume that  $(p_1 \ p_2 \ p_3) \in \mathcal{T}(T, U_{I_1 0} \oplus U'_{i0} \oplus P_{I_2}^*)$  satisfies  $(p_1 \ p_2 \ p_3) \begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \\ \mathbf{v}^* \end{pmatrix} = 0$ . We first show that there exists  $q_1 \in \mathcal{T}(T, U_{I_2 1})$  such that  $p_3 = q_1 \mathbf{b}^*$ . Since  $\mathbf{b}^*$  is right almost split in  $\text{add } T$  by Proposition 4.5 (a), we only have to show  $p_3 \in J_{\mathcal{T}}$ . We only have to consider the case  $T = P_l^*$  for  $l \in I_2$ . Then since  $p_3 \mathbf{v}^* = -p_1 \mathbf{a}u - p_2 \mathbf{c} \in J_{\text{add } T}^2$ , we have  $p_3 \in J_{\mathcal{T}}$ .

Then we have

$$\begin{aligned}
 ((p_1 - q_1 f_2) \mathbf{a} \quad p_2 - q_1 f'_2) \begin{pmatrix} \mathbf{u} \\ \mathbf{c} \end{pmatrix} &= (p_1 \ p_2) \begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \end{pmatrix} - q_1 (f_2 \ f'_2) \begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \end{pmatrix} \\
 &= (p_1 \ p_2) \begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \end{pmatrix} + q_1 \mathbf{b}^* \mathbf{v}^* \\
 &= (p_1 \ p_2) \begin{pmatrix} \mathbf{a}u \\ \mathbf{c} \end{pmatrix} + p_3 \mathbf{v}^* \\
 &= 0.
 \end{aligned}$$

Therefore, by Proposition 4.5 (c), there exists  $(g_1 \ g_2) \in \mathcal{T}(T, U'_{i1} \oplus P_{I_2})$  such that

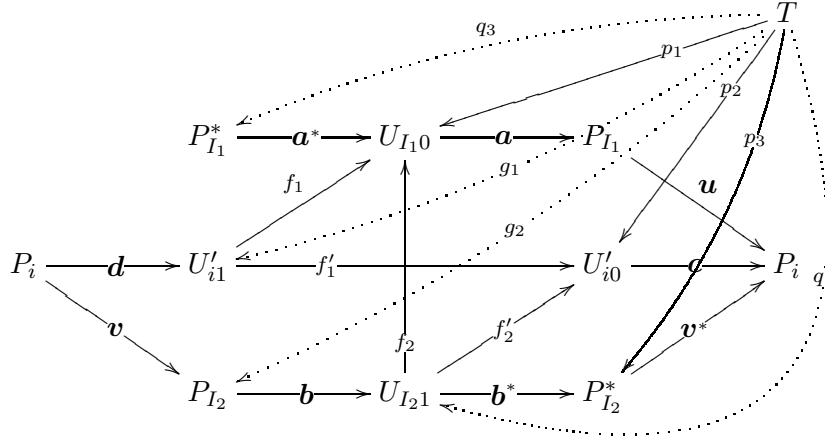
$$(g_1 \ g_2) \begin{pmatrix} f_1 \mathbf{a} & f'_1 \\ \mathbf{b} f_2 \mathbf{a} & \mathbf{b} f'_2 \end{pmatrix} = ((p_1 - q_1 f_2) \mathbf{a} \quad p_2 - q_1 f'_2).$$

Thus we have  $(p_1 \mathbf{a} \ p_2 \ p_3) = (g_1 \ g_2) \begin{pmatrix} f_1 \mathbf{a} & f'_1 & 0 \\ f_2 \mathbf{a} & f'_2 & \mathbf{b}^* \end{pmatrix}$ , where we put  $q_2 = q_1 + g_2 \mathbf{b}$ .

Moreover, since we have  $(p_1 - (g_1 f_1 + q_2 f_2)) \mathbf{a} = 0$ , there exists  $q_3 \in \mathcal{T}(T, P_{I_1}^*)$  such that  $q_3 \mathbf{a}^* = p_1 - (g_1 f_1 + q_2 f_2)$  by Proposition 4.5 (b). Thus we have

$$(p_1 \ p_2 \ p_3) = (q_3 \ g_1 \ g_2) \begin{pmatrix} \mathbf{a}^* & 0 & 0 \\ f_1 & f'_1 & 0 \\ f_2 & f'_2 & \mathbf{b}^* \end{pmatrix}.$$





□

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